Exponential Growth and Decay

Although the term is used loosely, and often incorrectly, many real-world phenomena grow or decay at an exponential rate. Population growth can be roughly modeled as an exponential growth process (populations of people, animals, bacteria, and more). Radioactive decay, used for carbon dating, is an example of exponential decay. Even heating and cooling, as we'll see, can be modeled by exponential functions.

What do we mean when we say population growth can be "roughly modeled as an exponential growth process?" We mean that we can find an exponential function $P(t)$ such that if we plug in a year, t, it will spit out the approximate population at that year. Furthermore, we can just assume that this exponential function has the form $P(t) = P_0 e^{kt}$, where P_0 and k are some constants that depend on the problem at hand.

Let's say we want to model the population growth of the world. We start with a few data points. In the year 1800, the world population was about 1 billion. The population was about 1.6 billion in the year 1900. It reached 2 billion in 1927, and had hit 3 billion by 1960. Let's plot those points.

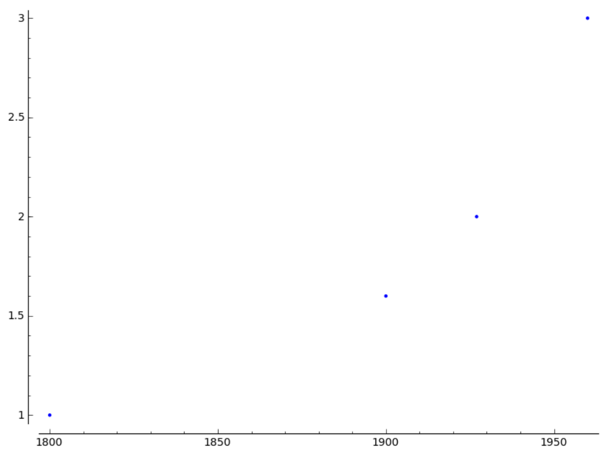

FIGURE 1.

Look at the shape of these points – it's quite a bit like an exponential curve! So, imagining we were studying population in the year 1960, we might try to find an exponential function that fits this data and will predict the population in the year 1980. Hopefully you're convinced that population growth can be modeled exponentially now, and why it might be useful (to then predict future populations!). Now let's get down to the problem-solving aspect of it.

If we decide that population growth can be modeled using a function of the form $P(t) = P_0 e^{kt}$, then what we have to do is find the right values of P_0 and k for our problem. Also, it turns out that we only need two data points to do this. We had four data points in our example above, so more than we needed – let's just say that we know the population in 1800 was 1 billion, and the population in 1960 was 3 billion. We'll find an exponential curve that goes through those two points. Rather than say $t = 1800$, we might consider for our purposes here that the year 1800 is "Year 0," when time starts, and let $t = 0$ represent 1800. Then $t = 160$ would represent 1960. Also, rather than writing our populations as 1,000,000,000 or 3,000,000,000, let's write them in billions to save space (the world is crowded enough!). Our two data points will give us two equations: $P(0) = 1$ and $P(160) = 3$. If we write those out using the fact that $P(t) = P_0 e^{kt}$, we get:

$$1 = P_0 e^{k \cdot 0}$$

and

$$3 = P_0 e^{k \cdot 160}$$

From these two equations, we can figure out what P_0 is and what k is. Take a closer look at the first equation:

$$\begin{aligned} 1 &= P_0 e^{k\cdot 0} \\ 1 &= P_0 e^0 \\ 1 &= P_0 \cdot 1 \\ 1 &= P_0 \end{aligned}$$

So this equation just simplified to tell us that P_0 must be 1! Now what about the other equation? If we know $P_0 = 1$, we can substitute that in:

$$\begin{aligned} 3 &= P_0 e^{k\cdot 160} \\ 3 &= 1 \cdot e^{160k} \\ \ln 3 &= \ln e^{160k} \\ \ln 3 &= 160k \\ \frac{\ln 3}{160} &= k \end{aligned}$$

Therefore, $k = \frac{\ln 3}{160}$, or if you happen to be using a calculator, $k \approx .00687$. So we're claiming that our world population can be modeled by the function $P(t) = e^{.00687t}$. How accurate is this model? We'll draw the function, along with our data points and some "future" data points on the graph below.

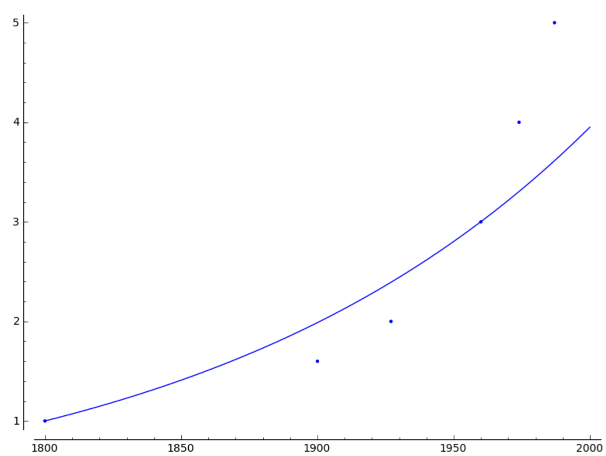

FIGURE 2.

As you can see, the curve goes exactly through the points $(1800, 1)$ and $(1960, 3)$, since those are the two points we used to create the model. It's not a perfect fit to the data, (those 1960 researchers would have underestimated population growth in future years), but could be made better if we used more data points to build our model. That's a bit more complicated, though, and will have to wait for another math class.

So to recap – to fit an exponential curve to data, you only need two data points. Those two data points will give you two equations, which you can simplify and solve to find your missing constants. Then, voila, you will have an exponential model!

Another quick example before you get problems of your own. Let's say you know that bacteria in a petri dish grows at an exponential rate. You put in some cells of bacteria, check back two hours later, and find there are 500 cells. Four hours after that, there are 1000 cells. How many cells were there when you started, and how many will there be after one day has passed?

Our two data points here are $P(2) = 500$ and $P(6) = 1000$, and we're wondering what $P(0)$ is and what $P(24)$ is. So let's find our two equations:

$$500 = P_0 e^{2k} \qquad \text{and} \qquad 1000 = P_0 e^{6k}$$

In this case, we're not as lucky as before, and can't just simplify one of the equations to find P_0 or k. But we can still solve and substitute, as follows. In the first equation, let's solve for P_0:

$$\begin{aligned} 500 &= P_0 e^{2k} \\ \frac{500}{e^{2k}} &= P_0 \end{aligned}$$

Then substitute that into the second equation, and simplify/solve:

$$\begin{aligned} 1000 &= P_0 e^{6k} \\ 1000 &= \frac{500}{e^{2k}} e^{6k} \\ \frac{1000}{500} &= \frac{e^{6k}}{e^{2k}} \\ 2 &= e^{4k} \\ \ln 2 &= 4k \\ \frac{\ln 2}{4} &= k \end{aligned}$$

Now that we've found k it will be easy to find P_0. Take that value of k and plug it back into either one of the equations and solve. Let's plug it into the first equation:

$$\begin{aligned} 500 &= P_0 e^{2 \cdot \frac{\ln 2}{4}} \\ 500 e^{-\frac{\ln 2}{2}} &= P_0 \end{aligned}$$

Therefore, our function is $P(t) = (500e^{-\frac{\ln 2}{2}})e^{\frac{\ln 2}{4}t}$. For a calculator-less class, we could answer our question exactly (but not quite as helpfully), by calculating $P(0)$ and $P(24)$:

$$P(0) = (500e^{-\frac{\ln 2}{2}})e^{\frac{\ln 2}{4} \cdot 0} \qquad \text{and} \qquad P(24) = (500e^{-\frac{\ln 2}{2}})e^{\frac{\ln 2}{4} \cdot 24}.$$

If we wanted more intuitive (but less exact) values, we could use a calculator to find an approximation, and we would get that $P(0) \approx 353.55$ and $P(24) \approx 22627.41$. Now try some. Note that decay works exactly the same, but your k values will end up being negative.

(1) A scientist isolates 2000 grams of a radioactive isotope. Five hours later, 1800 grams are left. Assuming this isotope decays exponentially, answer the following questions.
   (a) If the scientist returns in 5 more hours, how much of the isotope will remain?
   (b) What is the half-life of this isotope? (Half-life is the time it takes for the quantity of a substance to reduce to half its original value)

(2) The populations of ant hill A and and ant hill B are both growing exponentially. The following population data was recorded one day:

Ant hill	12:00pm	5:00pm
A	100	120
B	80	150

Approximate the time at which the population of ant hill B first exceed the population of ant hill A.

Now, by this time you may have noticed that P_0 is always the "initial amount" of something, the amount you start with. That makes sense in these types of problems, since $P(0) = P_0e^{k\cdot 0} = P_0e^0 = P_0$. However, there are other types of exponential models where it might not be so simple. A great example has to do with heating and cooling.

In a simplified form, Newton's law of heating and cooling says that the temperature of an object can be modeled as an exponential curve. For example, if you take a hot potato from the oven and put it into a 75 degree room, the potato's temperature will "decay" exponentially, falling quickly at first, but then cooling more and more slowly as the temperature of the potato gets closer to 75 degrees. Picture-wise, that means the potato's temperature will look something like this:

That is, instead of an exponential function having a horizontal asymptote at 0, it will have a horizontal asymptote at 75, since that's the temperature the potato will approach. Using our knowledge of graph transformations, we can see that the general formula for an exponential function with an asymptote at 75 is $P(t) = P_0e^{kt} + 75$. So when we generate our two equations from two data points, we want to use that general form.

For example, let's say the potato is 200 degrees in the oven. It's taken out into the 75-degree room, and 10 minutes later the potato is 170 degrees. When will it be 120 degrees? Our two data points here are $P(0) = 200$ and $P(10) = 170$. This gives us the two equations:

$$200 = P_0e^{k\cdot 0} + 75 \qquad \text{and} \qquad 170 = P_0e^{k\cdot 10} + 75$$

Solving the first one gives us $P_0 = 125$, which we can plug into the second one:

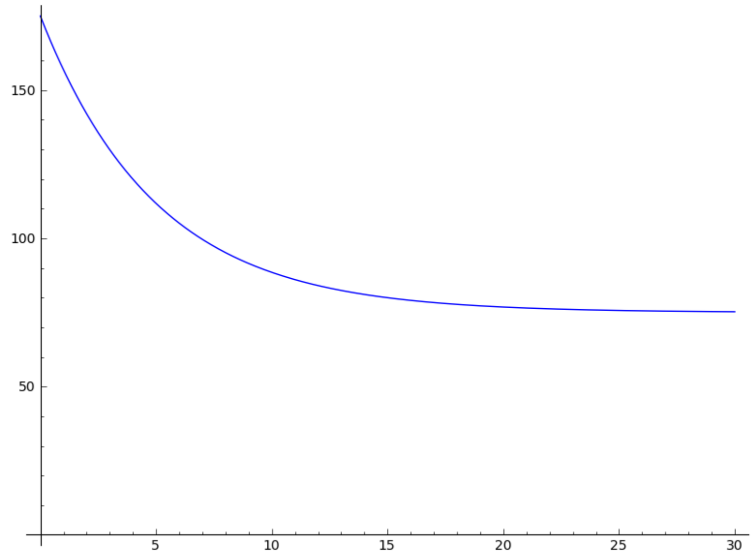

FIGURE 3.

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\begin{aligned}
170 &= 125e^{10k} + 75 \\
\frac{95}{125} &= e^{10k} \\
\ln\frac{95}{125} &= 10k \\
\frac{\ln\frac{95}{125}}{10} &= k
\end{aligned}
$$

So the temperature of the potato at any time is $P(t) = 125e^{\frac{\ln\frac{95}{125}}{10}t} + 75$. This is getting messy, so you'll want to be very careful in your computations. We can now answer the question of when it will be 120 degrees. We're wondering, for what value of t will $P(t) = 120$? So we have to solve

$$120 = 125e^{\frac{\ln\frac{95}{125}}{10}t} + 75$$

for t. You've practiced solving exponential equations now, so we won't include those steps. The final answer, however, is

$$t = \frac{10\ln(\frac{45}{125})}{\ln(\frac{95}{125})},$$

or about 37 minutes.

So, what's the difference between the potato problem and those previous problems? We just added a constant – instead of P_0e^{kt}, we made it $P_0e^{kt} + C$. Otherwise, it's all the same method!

(3) You boil a kettle of water and pour out a cup to make tea. Initially 100°C, it cools to 80°C in 10 minutes. How long will it take to reach a drinkable temperature of 65°C if the room is 25°C?

(4) It's Thanskgiving morning and you have a feast to prepare. You place your thawed, room temperature (70°F) turkey into a preheated oven (325°F) at 10 am. When you check at 11 am, the internal temperature of the turkey has reached 102°F. Will it reach a safe 180°F by the time your guests arrive at 4 pm?